SPACELIKE HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN THE STEADY STATE SPACE

ALMA L. ALBUJER AND LUIS J. ALÍAS

ABSTRACT. We consider complete spacelike hypersurfaces with constant mean curvature in the open region of de Sitter space known as the steady state space. We prove that if the hypersurface is bounded away from the infinity of the ambient space, then the mean curvature must be $H = 1$. Moreover, in the 2-dimensional case we obtain that the only complete spacelike surfaces with constant mean curvature which are bounded away from the infinity are the totally umbilical flat surfaces. We also derive some other consequences for hypersurfaces which are bounded away from the future infinity. Finally, using an isometrically equivalent model for the steady state space, we extend our results to a wider family of spacetimes.

1. The steady state space

Let $\mathbb{R}_1^{n+2}$ be the $(n + 2)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle , \rangle = -(dx_0)^2 + (dx_1)^2 + ... + (dx_{n+1})^2$$

where $(x_0, ..., x_{n+1})$ are the canonical coordinates of $\mathbb{R}^{n+2}$. As it is well known, the hyperquadric

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{R}_1^{n+2} : \langle x, x \rangle = 1\}$$

consisting of all unit spacelike vectors in $\mathbb{R}_1^{n+2}$ endowed with the induced metric from $\mathbb{R}_1^{n+2}$ is the de Sitter space. The de Sitter space is a complete simply connected $(n + 1)$-dimensional Lorentzian manifold with constant sectional curvature one. Therefore, $\mathbb{S}_1^{n+1}$ can be seen, in Lorentzian geometry, as the equivalent to the Euclidean sphere.

Take now a non-zero null vector $a \in \mathbb{R}_1^{n+2}$ past-pointing, that is, $\langle a, a \rangle = 0$ and $\langle a, e_0 \rangle > 0$ where $e_0 = (1, 0, ..., 0)$ and consider the following open region of the de Sitter space

$$\mathcal{H}^{n+1} = \{x \in \mathbb{S}_1^{n+1} : \langle x, a \rangle > 0\}.$$

The open region $\mathcal{H}^{n+1}$ forms the spacetime for the steady state model of the universe proposed by Bondi and Gold [4] and Hoyle [8], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but at all times (cf. [14, Section 14.8] and [7, Section 5.2]). The steady state space is a non-complete manifold, being only half of the de Sitter space and having as boundary the null hypersurface

$$L_0 = \{x \in \mathbb{S}_1^{n+1} : \langle x, a \rangle = 0\}.$$

The null hypersurface L_0 represents the past infinity of $\mathcal{H}^{n+1}$, usually denoted by $\mathcal{J}^-$, and the limit boundary

$$L_\infty = \{x \in \mathbb{S}_1^{n+1} : \langle x, a \rangle = \infty\}$$

represents its future infinity, usually denoted by $\mathcal{J}^+$.

2. Spacelike hypersurfaces in the steady state space

A smooth immersion $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ of an n-dimensional connected manifold Σ^n is said to be a spacelike hypersurface if ψ induces a Riemannian metric on Σ, which as usual is also denoted by $\langle,\rangle$. In that case, there exists a unique unitary timelike normal field N globally defined on Σ which is future-directed. Throughout this paper we will refer to N as the future-pointing Gauss map of Σ. The mean curvature function of a spacelike hypersurface Σ is defined as

$$H = -\frac{1}{n}\mathrm{tr}(A),$$

where A stands for the shape operator (or second fundamental form) of Σ with respect to its future-pointing Gauss map N. The choice of the sign $-$ in our definition of H is motivated by the fact that in that case the mean curvature vector is given by $\overrightarrow{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\overrightarrow{H}(p)$ is future-directed.

The steady state space has a special property, it admits a foliation by means of totally umbilical spacelike hypersurfaces

$$L_\tau = \{x \in \mathcal{H}^{n+1} : \langle x, a \rangle = \tau\}, \quad \tau > 0,$$

having constant mean curvature $H = 1$ with respect to their future-pointing Gauss map

$$N_\tau(x) = x - \frac{1}{\tau}a.$$

It is worth pointing out that each L_τ is isometric to the Euclidean space $\mathbb{R}^n$. Actually, if we identify $\mathbb{R}^n$ with the orthogonal complement in $\mathbb{R}_1^{n+2}$ of the Lorentzian plane spanned by a and e_0, then it is easy to see that the map $\phi_\tau : L_\tau \to \mathbb{R}^n$ given by

$$\phi_\tau(x) = x - \frac{\tau + \langle a, e_0 \rangle \langle x, e_0 \rangle}{\langle a, e_0 \rangle^2}a - \frac{\tau}{\langle a, e_0 \rangle}e_0$$

defines an isometry between L_τ and $\mathbb{R}^n$.

In this context, we will say that a spacelike hypersurface Σ in $\mathcal{H}^{n+1}$ is *bounded away from the future infinity* if there exists $\overline{\tau} > 0$ such that

$$\psi(\Sigma) \subset \{x \in \mathcal{H}^{n+1} : \langle x, a \rangle \leq \overline{\tau}\},$$

and we will say that it is *bounded away from the past infinity* if there exists $\underline{\tau} > 0$ such that

$$\psi(\Sigma) \subset \{x \in \mathcal{H}^{n+1} : \langle x, a \rangle \geq \underline{\tau}\}.$$

We will say that Σ is *bounded away from the infinity* if it is both bounded away from the past and the future infinity. In other words, Σ is bounded away from the infinity if there exist $0 < \underline{\tau} < \overline{\tau}$ such that $\psi(\Sigma)$ is contained in the slab bounded by $L_{\underline{\tau}}$ and $L_{\overline{\tau}}$.

Lemma 1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface. If Σ is bounded away from the future infinity of $\mathcal{H}^{n+1}$, then Σ is diffeomorphic to $\mathbb{R}^n$. In particular, there is no compact (without boundary) spacelike hypersurface in $\mathcal{H}^{n+1}$.*

Proof. We will see that Σ is diffeomorphic to $\mathbb{R}^n$ by showing that its orthogonal geodesic projection onto L_1 is a diffeomorphism. Observe that the steady state spacetime $\mathcal{H}^{n+1}$ can be globally parametrized by means of the diffeomorphism $\Phi : \mathbb{R} \times L_1 \to \mathcal{H}^{n+1}$ given by $\Phi(t, q) = \gamma_q(t)$, where

$$\gamma_q(t) = e^t \, q - \sinh t \, a$$

denotes the (future pointing) unitary geodesic orthogonal to L_1 through the point $q \in L_1$. Then, the orthogonal geodesic projection of $\mathcal{H}^{n+1}$ onto L_1 assigns to each $x \in \mathcal{H}^{n+1}$ the point $\Psi(x) \in L_1$ such that $x = \Phi(t, \Psi(x))$ for a certain t, and it is given by

$$\Psi(x) = \frac{1}{\langle x, a \rangle} x + \frac{1}{2}\left(1 - \frac{1}{\langle x, a \rangle^2}\right) a, \quad x \in \mathcal{H}^{n+1}.$$

Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a spacelike hypersurface. The orthogonal geodesic projection of Σ onto L_1 is the composition map $\Pi = \Psi \circ \psi : \Sigma \to L_1$ written as

$$\Pi = \frac{1}{\langle \psi, a \rangle} \psi + \frac{1}{2}\left(1 - \frac{1}{\langle \psi, a \rangle^2}\right) a.$$

A straightforward computation shows that

$$d\Pi_p(v) = \frac{1}{\langle \psi(p), a \rangle} d\psi_p(v) - \frac{\langle d\psi_p(v), a \rangle}{\langle \psi(p), a \rangle^2} \psi(p) + \frac{\langle d\psi_p(v), a \rangle}{\langle \psi(p), a \rangle^3} a$$

for every $p \in \Sigma$ and every tangent vector $v \in T_p\Sigma$. Therefore,

$$\begin{aligned}
(1) \qquad \langle d\Pi_p(v), d\Pi_p(v) \rangle_{\text{o}} &= \frac{1}{\langle \psi(p), a \rangle^2} \langle d\psi_p(v), d\psi_p(v) \rangle + \frac{\langle d\psi_p(v), a \rangle^2}{\langle \psi(p), a \rangle^4} \\
&\geq \frac{1}{\langle \psi(p), a \rangle^2} \langle d\psi_p(v), d\psi_p(v) \rangle,
\end{aligned}$$

that is,

$$\Pi^*(\langle , \rangle_{\text{o}}) \geq \frac{1}{\langle \psi, a \rangle^2} \langle , \rangle$$

where $\langle , \rangle_{\text{o}}$ denotes the flat Euclidean metric on L_1 and $\langle , \rangle$ denotes the Riemannian metric on Σ. Since we are assuming that Σ is bounded away from the future infinity of $\mathcal{H}^{n+1}$, it follows from here that

$$(2) \qquad\qquad \Pi^*(\langle , \rangle_{\text{o}}) \geq \frac{1}{\overline{\tau}^2} \langle , \rangle$$

for certain positive $\overline{\tau}$.

From (1) we get that Π is a local diffeomorphism. Since $\langle,\rangle$ is a complete Riemannian metric on Σ, the same holds for the homothetic metric $\widetilde{\langle,\rangle} = (1/\overline{\tau}^2)\langle,\rangle$. Then, (2) means that the map

$$\Pi : (\Sigma, \widetilde{\langle,\rangle}) \to (L_1, \langle,\rangle_{\mathrm{o}})$$

increases the distance, and hence Π is a covering map by [10, Chapter VIII, Lemma 8.1]. But L_1 being simply connected this means that Π is in fact a global diffeomorphism between Σ and L_1. $\qquad\square$

3. Spacelike hypersurfaces with constant mean curvature

For a spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$, consider the function $u \in \mathcal{C}^\infty(\Sigma)$ defined as $u(p) = \langle \psi(p), a \rangle$, $p \in \Sigma$. Observe that the gradient of u in Σ is

$$\nabla u = a^\top,$$

where $a^\top$ denotes the tangential component of a along Σ, that is,

$$(3) \qquad a = a^\top - \langle N, a \rangle N + \langle \psi, a \rangle \psi = \nabla u - \langle N, a \rangle N + \langle \psi, a \rangle \psi.$$

Using Gauss and Weingarten formulae, we easily obtain

$$\nabla_X \nabla u = -\langle N, a \rangle AX - uX$$

for every $X \in T\Sigma$. Therefore, the Laplacian of the function u on Σ is given by

$$(4) \qquad \Delta u = nH\langle N, a \rangle - nu.$$

From (3), it is also easy to see that

$$(5) \qquad \|\nabla u\|^2 = \langle N, a \rangle^2 - u^2$$

where $\|\cdot\|$ denotes the norm of a vector field on Σ. Recall here that a is a past-pointing null vector and N is future-pointing, so that $\langle N, a \rangle > 0$ on Σ.

In order to prove our main results, we will make use of the well-known Omori-Yau maximum principle [15], recalled next.

Lemma 2. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If $u \in \mathcal{C}^\infty(M)$ is bounded from above on M then there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in M such that*

$$\lim_{k\to\infty} u(p_k) = \sup_M u, \quad \|\nabla u(p_k)\| < 1/k \quad \text{and} \quad \Delta u(p_k) < 1/k.$$

Now we are ready to state the following result.

Theorem 3. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with constant mean curvature H. If Σ is bounded away from the infinity of $\mathcal{H}^{n+1}$, then $H = 1$ necessarily. Moreover, in the 2-dimensional case, there exists τ such that $\Sigma^2 = L_\tau$.*

Proof. The Gauss equation of Σ in $\mathcal{H}^{n+1}$ describes the curvature of Σ, denoted by R, in terms of its shape operator, and it is given by

$$\langle R(X,Y)X, Y \rangle = \langle X, X \rangle\langle Y, Y \rangle - \langle X, Y \rangle^2 - \langle AX, X \rangle\langle AY, Y \rangle + \langle AX, Y \rangle^2$$

being $X, Y \in T\Sigma$. Taking traces here, we obtain that for every $X \in T\Sigma$, $\|X\| = 1$,

$$(6) \qquad \mathrm{Ric}(X,X) \;=\; n - 1 + nH\langle AX, X \rangle + \langle AX, AX \rangle$$

$$\;=\; n - 1 + \left\| AX + \frac{nH}{2}X \right\|^2 - \frac{n^2 H^2}{4} \geq n - 1 - \frac{n^2 H^2}{4},$$

where Ric stands for the Ricci curvature of Σ. Thus, the Ricci curvature of Σ is bounded from below by the constant $n - 1 - n^2 H^2/4$. By assumption the function u is bounded (from above and from below). Therefore, applying Lemma 2 to the function u, we know that there exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ in Σ such that

$$\lim_{k\to\infty} u(p_k) = \sup_{\Sigma} u < +\infty, \tag{7}$$

$$\|\nabla u(p_k)\|^2 = \langle N(p_k), a\rangle^2 - u^2(p_k) < 1/k^2 \tag{8}$$

and

$$\Delta u(p_k) = n(H\langle N(p_k), a\rangle - u(p_k)) < 1/k.$$

From the last equation, taking into account that $\langle N, a\rangle > 0$ on Σ, we obtain that

$$H < \frac{u(p_k)}{\langle N(p_k), a\rangle} + \frac{1}{n\langle N(p_k), a\rangle k}. \tag{9}$$

On the other hand, taking limits in (8) and using (7) we also get that

$$\lim_{k\to\infty} \langle N(p_k), a\rangle = \lim_{k\to\infty} u(p_k) = \sup_{\Sigma} u.$$

Therefore, taking limits in (9) we conclude that $H \leq 1$.

In an similar way, we may apply Lemma 2 to the function $-u$, obtaining another sequence $\{q_k\}_{k\in\mathbb{N}}$ in Σ such that

$$\lim_{k\to\infty} u(q_k) = \inf_{\Sigma} u > 0,$$

$$\|\nabla u(q_k)\|^2 = \langle N(q_k), a\rangle^2 - u^2(q_k) < 1/k^2$$

and

$$\Delta u(q_k) = n(H\langle N(q_k), a\rangle - u(q_k)) > -1/k.$$

Therefore, we get now that

$$H > \frac{u(q_k)}{\langle N(q_k), a\rangle} - \frac{1}{n\langle N(q_k), a\rangle k}$$

and taking limits again we conclude that $H \geq 1$. Summing up $H = 1$. Moreover, when $n = 2$ we know by a result due to Akutagawa [2] and Ramanathan [13] that Σ must be a totally umbilical surface of the foliation L_τ. For the sake of completeness, we give here a different and direct proof of this assertion in our situation.

Hence, assume that $n = 2$. Since $H = 1$, (6) implies that the Gaussian curvature of Σ is non-negative. Then Σ is a complete Riemannian surface with non-negative Gaussian curvature, and by a classical result due to Ahlfors [1] and Blanc-Fiala-Huber [9], Σ is parabolic, in the sense that any subharmonic function bounded from above on the surface must be constant. By (5) and from the fact that $\langle N, a\rangle > 0$ we obtain that

$$\langle N, a\rangle - u \geq 0$$

on Σ. Then, (4) implies that $\Delta u \geq 0$. Therefore u is a subharmonic function bounded from above on the parabolic surface Σ, and hence it must be constant, being Σ one of the totally umbilical spacelike surfaces L_τ. $\qquad\square$

Corollary 4. *The only complete spacelike surfaces with constant mean curvature in $\mathcal{H}^3$ which are bounded away from the infinity are the totally umbilical surfaces L_τ.*

As a consequence of our Lemma 1 and the proof of Theorem 3 we can also state the following result.

Theorem 5. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with constant mean curvature H. If Σ is bounded away from the future infinity of $\mathcal{H}^{n+1}$ and has future-pointing mean curvature vector, then $2\sqrt{n-1}/n \leq H \leq 1$. Moreover, in the 2-dimensional case, there exists τ such that $\Sigma^2 = L_\tau$.*

Proof. In this case the function u is bounded only from above. Therefore, by the first part of the proof of Theorem 3 we obtain that $H \leq 1$. On the other hand, estimate (6) implies that $H^2 \geq 4(n-1)/n^2$. Otherwise by Bonnet-Myers' theorem we would get that Σ is compact, which is not possible by Lemma 1. But the mean curvature vector field being future-pointing yields $H \geq 2\sqrt{n-1}/n$. Summing up, $2\sqrt{n-1}/n \leq H \leq 1$. When $n = 2$ this gives $H = 1$ and then Σ must be one of the totally umbilical spacelike surfaces L_τ. $\square$

Corollary 6. *The only complete spacelike surfaces with constant mean curvature in $\mathcal{H}^3$ which are bounded away from the future infinity and have future-pointing mean curvature vector are the totally umbilical surfaces L_τ.*

It is worth pointing out that Corollary 6 is no longer true for surfaces which are bounded away from the past infinity of $\mathcal{H}^3$. Actually, in [11, Corollary 12] Montiel constructs complete spacelike hypersurfaces in $\mathcal{H}^{n+1}$ with constant mean curvature $H > 1$ which are bounded away from the past infinity. On the other hand, in relation to our Corollaries 4 and 6, we also refer the reader to Theorem 4.4 and Theorem 4.5 in [5] where Caminha and de Lima have recently found another interesting characterizations of the totally umbilical surfaces L_τ in $\mathcal{H}^3$.

4. AN ISOMETRICALLY EQUIVALENT MODEL. STEADY STATE TYPE SPACETIMES

The steady state space can be expressed in an isometrically equivalent way as the generalized Robertson-Walker spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^n$. That is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric tensor

$$\langle , \rangle = -dt^2 + e^{2t}(dx_1^2 + ... + dx_n^2),$$

$(t, x_1, ...x_n)$ being the canonical coordinates in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. To see it, take $b \in \mathbb{R}_1^{n+2}$ another null vector such that $\langle a, b \rangle = 1$ and let $\Phi : \mathcal{H}^{n+1} \to -\mathbb{R} \times_{e^t} \mathbb{R}^n$ be the map given by

$$(10) \qquad \Phi(x) = \left(\log(\langle x, a \rangle), \frac{x - \langle x, a \rangle b - \langle x, b \rangle a}{\langle x, a \rangle} \right).$$

Then it can easily be checked that

$$(d\Phi)_x(v) = \left(\frac{\langle v, a \rangle}{\langle x, a \rangle}, \frac{(v - \langle v, a \rangle b - \langle v, b \rangle a)\langle x, a \rangle - \langle v, a \rangle(x - \langle x, a \rangle b - \langle x, b \rangle a)}{\langle x, a \rangle^2} \right)$$

and that Φ is an isometry between both spaces which conserves time orientation.

Therefore, a natural extension of the steady state space consists on considering the wider family of Lorentzian manifolds defined as follows. Let M^n be a connected n-dimensional Riemannian manifold and consider the product manifold $\mathbb{R} \times M^n$ endowed with the Lorentzian metric tensor

$$\langle , \rangle = -\pi_{\mathbb{R}}^*(dt^2) + e^{2t}\pi_M^*(\langle , \rangle_M)$$

where $\pi_{\mathbb{R}}$ and π_M denote the projections from $\mathbb{R} \times M^n$ onto each factor, and $\langle,\rangle_M$ is the Riemannian metric on M. For simplicity, we will write

$$\langle,\rangle = -dt^2 + e^{2t}\langle,\rangle_M.$$

We will denote by $-\mathbb{R} \times_{e^t} M^n$ the $(n+1)$-dimensional product manifold $\mathbb{R} \times M^n$ endowed with that Lorentzian metric, and we will refer to them as steady state type spacetimes. For instance, when M^n is the flat n-torus we get the de Sitter cusp as defined in [6].

5. Spacelike hypersurfaces in a steady state type spacetime

Let $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a spacelike hypersurface. Observe that

$$\partial_t = (\partial/\partial t)_{(t,x)}, \ t \in \mathbb{R}, \ x \in M,$$

is a unitary timelike vector field globally defined on the ambient spacetime which determines a time-orientation on $-\mathbb{R} \times_{e^t} M^n$. Then, the future-pointing Gauss map of Σ is the unique unitary timelike normal vector field N globally defined on Σ in the same time-orientation as ∂_t. Thus, we have

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on} \quad \Sigma.$$

We will denote by $\Theta : \Sigma \to (-\infty, -1]$ the smooth function on Σ given by $\Theta = \langle N, \partial_t \rangle$. Observe that the function Θ measures the hyperbolic angle θ between the future-pointing vector fields N and ∂_t along Σ. Indeed, they are related by $\cosh \theta = -\Theta$.

For a spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$, we define the height function of Σ, denoted by h, as the projection of Σ onto $\mathbb{R}$, that is, $h \in \mathcal{C}^\infty(\Sigma)$ is the smooth function given by $h = \pi_{\mathbb{R}} \circ \psi$. In the particular case when $M^n = \mathbb{R}^n$, by (10) the height function of Σ can be written in terms of the function u as

$$h(p) = \log u(\Phi^{-1}(\psi(p))), \quad p \in \Sigma.$$

Therefore, the umbilical hypersurfaces L_τ, for which the function u takes the constant value $u = \tau$, correspond to the slices $\{\log(\tau)\} \times \mathbb{R}^n$. Actually, for a general steady state type spacetime $-\mathbb{R} \times_{e^t} M^n$, each leaf of the foliation $\{t\} \times M$ (called here a *slice*) of $-\mathbb{R} \times_{e^t} M^n$ is a totally umbilical spacelike hypersurface with constant mean curvature $H = 1$.

According to the terminology introduced for hypersurfaces in the steady state space, we will say that a spacelike hypersurface Σ in $-\mathbb{R} \times_{e^t} M^n$ is bounded away from the future infinity if there exists $\overline{t} \in \mathbb{R}$ such that

$$\psi(\Sigma) \subset \{(t,x) \in -\mathbb{R} \times_{e^t} M^n : t \leq \overline{t}\},$$

and we will say that Σ is bounded away from the past infinity if there exists $\underline{t} \in \mathbb{R}$ such that

$$\psi(\Sigma) \subset \{(t,x) \in -\mathbb{R} \times_{e^t} M^n : t \geq \underline{t}\}.$$

We will say that Σ is bounded away from the infinity if it is bounded away both from the past and the future infinity. The following result is the natural extension of our Lemma 1.

Lemma 7. *Let M^n be a Riemannian manifold. If $-\mathbb{R} \times_{e^t} M^n$ admits a complete spacelike hypersurface Σ which is bounded away from the future infinity, then M is necessarily complete and the projection $\Pi = \pi_M \circ \psi : \Sigma \to M$ is a covering map.*

Proof. The proof follows the ideas of [3, Lemma 3.1]. Let $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete spacelike hypersurface and let $\Pi = \pi_M \circ \psi : \Sigma \to M$ denote its projection on M. Observe that

$$\langle v, v \rangle = -\langle d\psi_p(v), \partial_t \rangle^2 + e^{2h(p)} \langle d\Pi_p(v), d\Pi_p(v) \rangle_M \leq e^{2h(p)} \langle d\Pi_p(v), d\Pi_p(v) \rangle_M$$

for every $p \in \Sigma$ and $v \in T_p\Sigma$. Thus, $\Pi^*(\langle, \rangle_M) \geq (1/e^{2h})\langle, \rangle$. As Σ is bounded away from the future infinity, it follows from here that

$$\Pi^*(\langle, \rangle_M) \geq \frac{1}{e^{2\bar{t}}}\langle, \rangle = \widetilde{\langle, \rangle}$$

for a certain real number $\bar{t}$. Then reasoning as in the proof of Lemma 1, we get that $\Pi : (\Sigma, \widetilde{\langle, \rangle}) \to (M, \langle, \rangle_M)$ is a local diffeomorphism which increases the distance. The proof finishes recalling that if a map, from a connected complete Riemannian manifold M_1 into another connected Riemannian manifold M_2 of the same dimension, increases the distance, then it is a covering map and M_2 is complete [10, Chapter VIII, Lemma 8.1]. $\qquad\square$

Let $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a spacelike hypersurface with height function h. Observe that the gradient of $\pi_\mathbb{R}$ on $-\mathbb{R} \times_{e^t} M^n$ is

$$\overline{\nabla}\pi_\mathbb{R} = -\langle \overline{\nabla}\pi_\mathbb{R}, \partial_t \rangle \partial_t = -\partial_t.$$

Therefore, the gradient of h on Σ is

$$\nabla h = (\overline{\nabla}\pi_\mathbb{R})^\top = -\partial_t^\top,$$

where $\partial_t^\top$ denotes the tangential component of ∂_t along Σ, that is,

$$\partial_t = \partial_t^\top - \Theta N.$$

In particular,

$$(11) \qquad\qquad\qquad \|\nabla h\|^2 = \Theta^2 - 1.$$

Let Y denote the timelike vector field on $-\mathbb{R} \times_{e^t} M^n$ given by

$$Y(t, x) = e^t(\partial/\partial t)_{(t,x)}, \qquad (t, x) \in -\mathbb{R} \times_{e^t} M^n.$$

From the relation between $\overline{\nabla}$, the Levi-Civita connection of the ambient space $-\mathbb{R} \times_{e^t} M^n$, and the Levi-Civita connection of M, we get that

$$(12) \qquad\qquad\qquad \overline{\nabla}_Z Y = e^t Z$$

for any vector field Z on $-\mathbb{R} \times_{e^t} M^n$. That is, Y is a globally defined closed conformal vector field on $-\mathbb{R} \times_{e^t} M^n$. Writing $\partial_t = -\nabla h - \Theta N$ along the hypersurface Σ and using Gauss and Weingarten formulae, it is not difficult to get from (12) that

$$e^h X = \langle \nabla e^h, X \rangle \partial_t - e^h \nabla_X \nabla h + e^h \langle AX, \nabla h \rangle N + e^h \Theta AX - e^h X(\Theta) N$$

for every $X \in T\Sigma$. Therefore,

$$\nabla_X \nabla h = \Theta AX - X - \langle \nabla h, X \rangle \nabla h$$

for every $X \in T\Sigma$, and the Laplacian on Σ of the height function is given by

$$(13) \qquad\qquad\qquad \Delta h = -nH\Theta - (n + \|\nabla h\|^2).$$

Now we are ready to state the main result of this section, which is the natural extension of our Theorem 3 to the wider family of steady state type spacetimes.

Theorem 8. *Let M^n be a (necessarily complete) Riemannian manifold with non-negative sectional curvature and let $\psi : \Sigma^n \rightarrow -\mathbb{R} \times_{e^t} M^n$ be a complete spacelike hypersurface with constant mean curvature H. If Σ is bounded away from the infinity then $H = 1$. Moreover, when $n = 2$, the surface Σ is necessarily a slice $\{t\} \times M$.*

Proof. The Gauss equation of Σ is given by

$$\langle R(X,Y)X,Y \rangle = \langle \overline{R}(X,Y)X,Y \rangle - \langle AX,X \rangle \langle AY,Y \rangle + \langle AX,Y \rangle^2$$

for every $X,Y \in T\Sigma$, where R and $\overline{R}$ stand for the curvature tensor of Σ and $-\mathbb{R} \times_{e^t} M^n$, respectively. Taking traces here, we get that for every $X \in T\Sigma$, $\|X\| = 1$,

$$
(14) \qquad
\begin{aligned}
\mathrm{Ric}(X,X) &= \sum_{i=1}^{m} \langle \overline{R}(X,E_i)X,E_i \rangle + nH\langle AX,X \rangle + \langle AX,AX \rangle \\
&\geq \sum_{i=1}^{m} \langle \overline{R}(X,E_i)X,E_i \rangle - \frac{n^2 H^2}{4},
\end{aligned}
$$

where Ric stands for the Ricci curvature of Σ. On the other hand, taking into account the relationship between the curvature tensor of $-\mathbb{R} \times_{e^t} M^n$ and the curvature tensor R_M of M (as well as the derivatives of the warping function) [12, Proposition 42], we also have that

$$(15) \qquad \langle \overline{R}(X,E_i)X,E_i \rangle = e^{2h}\langle R_M(X^*,E_i^*)X^*,E_i^* \rangle_M - \langle X,E_i \rangle^2 + 1,$$

for $1 \leq i \leq n$, where $X^* = (\pi_M)_*(X)$ denotes the projection of the vector field $X \in T\Sigma$ onto M, that is, $X = X^* - \langle X, \partial_t \rangle \partial_t$. Denoting by $K_M(X^* \wedge E_i^*)$ the sectional curvature in M of the 2-plane generated by X^* and E_i^*, we have that

$$\langle R_M(X^*,E_i^*)X^*,E_i^* \rangle_M = K_M(X^* \wedge E_i^*)\|X^* \wedge E_i^*\|_M^2,$$

where, as usual,

$$\|X^* \wedge E_i^*\|_M^2 = \langle X^*,X^* \rangle_M \langle E_i^*,E_i^* \rangle_M - \langle X^*,E_i^* \rangle_M^2.$$

Using this into (15), we get from (14)

$$
(16) \qquad
\begin{aligned}
\mathrm{Ric}(X,X) &\geq e^{2h}\sum_{i=1}^{n} K_M(X^* \wedge E_i^*)\|X^* \wedge E_i^*\|_M^2 + n - 1 - \frac{n^2 H^2}{4} \\
&\geq n - 1 - \frac{n^2 H^2}{4}.
\end{aligned}
$$

Consequently, the assumption on the sectional curvature of M implies that the Ricci curvature tensor of Σ is bounded from below. The proof that $H = 1$ follows now by applying Lemma 2 to the height function. Actually, since h is bounded from above we know that there exists a sequence $\{p_k\}_{k \in \mathbb{N}}$ in Σ such that

$$\lim_{k \to \infty} h(p_k) = \sup_{\Sigma} h < +\infty, \quad \|\nabla h(p_k)\|^2 = \Theta^2(p_k) - 1 < 1/k^2$$

and

$$\Delta h(p_k) = -nH\Theta(p_k) - (n + \|\nabla h(p_k)\|^2) < 1/k.$$

Therefore $\lim_{k \to \infty} \Theta(p_k) = -1$, and taking limits in the last equation we get $H \leq 1$.

In a similar way, by applying Lemma 2 to the function $-h$ we obtain another sequence $\{q_k\}_{k \in \mathbb{N}}$ in Σ such that

$$\lim_{k \to \infty} h(q_k) = \inf_\Sigma h > -\infty, \quad \|\nabla h(q_k)\|^2 = \Theta^2(q_k) - 1 < 1/k^2$$

and

$$\Delta h(q_k) = -nH\Theta(q_k) - (n + \|\nabla h(q_k)\|^2) > -1/k.$$

Thus $\lim_{k \to \infty} \Theta(q_k) = -1$ again, and taking limits in the last equation we get $H \geq 1$. As a consequence, $H = 1$.

Consider now the 2-dimensional case. Since $H = 1$, (16) implies that the Gaussian curvature of Σ is non-negative and, therefore, Σ is a parabolic surface. Moreover, (13), jointly with (11), implies that

$$\Delta h = -2\Theta - (2 + \|\nabla h\|^2) = -2\Theta - (1 + \Theta^2) = -(\Theta + 1)^2 \leq 0.$$

Therefore, h is a superharmonic function bounded from below on the parabolic surface Σ and hence it must be constant. $\qquad\square$

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Departamento de Matemáticas, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain

E-mail address: albujer@um.es

Departamento de Matemáticas, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain

E-mail address: ljalias@um.es